Godel's Incompleteness Theorems

and their Mathematical Ramifications

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Background

- In the 1880's Georg Cantor developed his *Theory of Sets*
    - ...but it led to a pandora's box of paradoxes...

- Russell's Paradox:
    - We can break types of sets into two groups:
        - run-of-the-mill sets: sets who are not members of themselves (i.e. a set of students)
        - self-swallowing sets: sets who do contain themselves members (i.e. the set of all things except students)
    - We can construct a set R:
        - R: the set of all run-of-the-mill sets
    - Is R run-of-the-mill or self-swallowing?
        - Neither, it is both! Paradox...

Attempts to Cope

- Russell and Whitehead's *Principia Mathematica*□ (1913)
  - Goals:
    - Derive all of mathematics from a well-defined set of axioms and inference rules from logic
    - Be self-consistent
- Hilbert's Program included:
  - **Formalize** all of mathematics
  - Show that mathematics is **Complete**
  - Show that mathematics is **Consistent**
  - Show that all mathematical statements are **Decidable**

But along comes Gödel...

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- A Sketch of a proof:
  - **Key Insight**: A statement of number theory could be *about* a statement of number theory.
  - **Goal**: Construct a true but unprovable mathematical sentence within number theory
  - **Process**: Using Gödel-numbering, symbols are assigned representative numbers, so mathematical statements acquire Gödel-numbers which can be used in other mathematical statements are a reference.

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- A Sketch of a proof cont...:
  - **Process**: Now within number theory, statements have the ability to "talk" about other statements.
  - **Construct** a statement which says:
    "This statement about number theory does not have any proof."
  - **Conclusion**: We have a statement within number theory that is true yet unprovable. ■

Ramifications

- Of Russell and Whitehead's goals in the *Principia Mathematica* to be Complete and Consistent,
    - Completeness has been proven impossible
- Of Formalization, Completeness, Consistency, and Decidability in Hilbert's Program,
    - Formalization, Completeness, and Decidability have been proven impossible
- Still some hope left in consistency!

There's more...

- Second Incompleteness Theorem:
  - Any formal system that is interesting enough to formulate its own consistency can prove its own consistency *if and only if* it is inconsistent (WolframMathWorld).

- Sketch of Proof:
  - Left to the curious to seek out
  - Basic Idea: Formalize the proof of the first incompleteness theorem within this formal system itself... $\blacksquare$

Ramifications cont...

- Of the last hope for Russell, Whitehead, and Hilbert: Consistency, Gödel has dismissed it as well.
- If a axiomatic system is consistent, it cannot prove that it is so.

Conclusions

- Godel sent a shock through the mathematical world with his incompleteness theorems.

- In summary,
  - If an axiomatic system is consistent, it is incomplete.
  - If it is consistent, it cannot prove that it is so.
  - And it follows, if the system is complete, then it is inconsistent.

Sources

Hoftstadter, Douglas R. *Gödel, Escher, Bach: an Eternal Golden Braid.* New York: Basic Books, Inc., 1979. Print

Weisstein, Eric W. "Gödel's Incompleteness Theorem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/GoedelsIncompletenessTheorem.html